# RGE Analysis of Neutrino Mass Matrix

based on T.K. Kuo, J. Pantaleone, G.-H. Wu hep-ph/0104131

- parametrization of 2×2 complex $m_\nu$
- RGE for $m_\nu$ and exact solution
- RGE Analysis:
  - Phase diagrams (flow diagrams)
  - Fixed points
  - RGE invariants.
- physical implications for maximal mixing
- generalizations and summary

Guo-Hong Wu
U. of Oregon

motivation:

- Condition for RGE-induced large $\nu$ mixing
- Stability of solution
- role of Majorana phase of $\nu$

Focus on:

- $2\times2$ complex, symmetric $m_\nu$
- SM and MSSM

May be generalized:

- other models: type III 2HDM, ...
- 3 $\nu$'s.

For 2 flavors, $(\nu_\mu, \nu_\tau)$, the RGE for $m_\nu$ in SM and MSSM:

$$\frac{d}{dt} m_\nu = -(\kappa m_\nu + m_\nu P + P^T m_\nu)$$

$$t = \frac{1}{16\pi^2} \ln \frac{\mu}{M_X}$$

$\kappa$: number that depends on $g_i$, $y_i$

$$P = P^T \approx x(1-\sigma_3)$$

$$x = \begin{cases} y_\tau^2/4 & \text{SM} \\ -\frac{1}{2}\tilde{y}_\tau^2 = -\frac{1}{2}\left(\frac{\sqrt{2}\, m_\tau}{v \cos\beta}\right)^2 & \text{MSSM} \end{cases}$$

Will ignore the t-dependence of $\kappa$ and $x$ to 1st approximation

Solution:

$$m_\nu(t) = e^{-\kappa' t}\, e^{\xi \sigma_3}\, m_\nu(0)\, e^{\xi \sigma_3}$$

$$\kappa' = \kappa + 2\chi$$
$$\xi = \chi t$$

For the study of $\nu$ mixing, convenient to separate out the overall mass scale

$$m_\nu = \sqrt{m_1 m_2}\, M$$

$M$: dimensionless, $\det M = 1$,
contains mixing angle, complex mass ratio.

$$\begin{cases} \sqrt{m_1(t) m_2(t)} = e^{-\kappa' t} \sqrt{m_1(0) m_2(0)} \\ M(t) = e^{\xi \sigma_3} M(0)\, e^{\xi \sigma_3} \end{cases}$$

$|\xi| \ll 1$

parametrization of M.

$$\begin{cases} M(t) = U(t)\, e^{-2\eta\sigma_3}\, U(t)^T \\ U(t) = e^{-i\alpha\sigma_3}\, e^{-i\theta\sigma_2}\, e^{-i\phi\sigma_3} \end{cases}$$

$\eta = \frac{1}{4}\ln\frac{m_2}{m_1}$  $m_1 > 0,\ m_2 > 0$

$\theta$: physical mixing angle

$4\phi$. relative Majorana phase

$\alpha$: unphysical

In explicit form,

$$M = \begin{pmatrix} e^{-2i\alpha}(\mathrm{ch}\,2\bar\eta - C_{2\theta}\,\mathrm{sh}\,2\bar\eta) & -S_{2\theta}\,\mathrm{sh}\,2\bar\eta \\ -S_{2\theta}\,\mathrm{sh}\,2\bar\eta & e^{2i\alpha}(\mathrm{ch}\,2\bar\eta + C_{2\theta}\,\mathrm{sh}\,2\bar\eta) \end{pmatrix}$$

$\bar\eta \equiv \eta + i\phi$

under RGE evolution,

$$M(0) \to M(t) = e^{\xi\sigma_3} M(0)\, e^{\xi\sigma_3}$$

$\Rightarrow$ $M_{12}$ RGE invariant

$$S_{2\theta}\, \mathrm{sh}\, 2\bar{\eta} = S_{2\theta_0}\, \mathrm{sh}\, 2\bar{\eta}_0$$

$\Rightarrow$ exact solution for $(\alpha, \theta, \eta, \varphi)$

$$\begin{cases} \tan 2\theta = \dfrac{S_{2\theta_0} / (C_{2\Delta\alpha}\, \mathrm{ch}\, 2\xi)}{C_{2\theta_0} - \Sigma_R \tanh 2\xi + \Sigma_I \tan 2\Delta\alpha} \\[2ex] \tan 2\Delta\alpha = \dfrac{\Sigma_I}{C_{2\theta_0} - \Sigma_R \coth 2\xi} \\[2ex] \Sigma_R + i\Sigma_I \equiv \coth 2\bar{\eta}_0 \\[2ex] \Delta\alpha = \alpha - \alpha_0 \end{cases}$$

"+" $\quad \mathrm{sh}\, 2\bar{\eta} = S_{2\theta_0}\, \mathrm{sh}\, 2\bar{\eta}_0 / S_{2\theta}$

- real matrix. $\Sigma_I = 0$, $\Delta\alpha = 0$

RGE.

in terms of M:

$$\frac{d}{dt}M = x\{M, \sigma_3\}$$

in terms of $(\alpha, \theta, \varphi, \eta)$:

$$\begin{cases} \dfrac{d\alpha}{dt} = x\,\dfrac{S_{4\phi}}{\mathrm{sh}4\eta} \\ \dfrac{d\eta}{dt} = -x\,C_{2\theta} \\ \dfrac{d\phi}{dt} = -x\,C_{2\theta}\,S_{4\phi}/\mathrm{sh}4\eta \\ \dfrac{d\theta}{dt} = x\,S_{2\theta}\left[C_{2\phi}^2 \coth 2\eta + S_{2\phi}^2 \tanh 2\eta\right] \end{cases}$$

① $\Rightarrow \frac{d}{dt}(S_{2\theta}\,\mathrm{sh}2\eta) = 0$ RGE inv.

② $(\eta, \theta, \varphi)$ indep. of $\alpha$

③ $\eta \to 0$ (degenerate), $\dot{\theta} \to \infty$, $\dot{\phi} \to \infty$

large RGE effect even for $x \ll 1$

④ real mass matrix,

$\phi = 0, \frac{\pi}{4}$

$\frac{d\phi}{dt} = 0 \quad \frac{d\alpha}{dt} = 0$

RGE preserves reality of M

$$\begin{cases} \frac{d\eta}{dt} = -\chi C_{2\theta} \\ \frac{d\theta}{dt} = \begin{cases} \chi S_{2\theta} \frac{m_2+m_1}{m_2-m_1} & (\phi=0, \text{ same sign masses}) \\ \chi S_{2\theta} \frac{m_2-m_1}{m_2+m_1} & (\phi=\frac{\pi}{4}, \text{ opposite sign}) \end{cases} \end{cases}$$

- large RGE effect possible for $\phi = 0$.

  but not for opposite sign masses ($\phi = \frac{\pi}{4}$)

⑤ symmetries of RGE/mass matrix M

$\theta \to \frac{\pi}{2} - \theta, \quad \eta \to -\eta, \quad \phi \to -\phi \quad \alpha \to \alpha$

$(m_2 \leftrightarrow m_1)$

$\Rightarrow$ small mixing w/ hierarchical $m_1 \ll m_2$

$\Leftrightarrow \simeq 90°$ mixing w/ inverted hierarchy

⑥ Fixed points

$\theta = 0, \quad \eta = \pm\infty \quad (\text{or: } \frac{m_1}{m_2} = 0, \infty)$

$\theta = \frac{\pi}{2}, \quad \eta = \pm\infty \qquad (\text{symmetry}).$

$\theta = \frac{\pi}{4}, \quad \eta = 0, \quad \phi = \frac{\pi}{4} \quad (m_1 = m_2, \text{ opposite sign})$

For clarity of phase diagram presentation, want to confine FP's to a finite region via change of variables:

$$z = \coth 2\bar{\eta} = \frac{m_2 e^{2i\phi} + m_1 e^{-2i\phi}}{m_2 e^{2i\phi} - m_1 e^{-2i\phi}}$$

$$\begin{cases} \dfrac{dz}{dt} = 4\chi\, C_{2\theta}\, \dfrac{(-1+z^2)\, z^*}{z + z^*} \\[2ex] \dfrac{d\theta}{dt} = 2\chi\, S_{2\theta}\, \dfrac{|z|^2}{z + z^*} \end{cases}$$

symmetry: $\quad \theta \to \frac{\pi}{2} - \theta \qquad z \to -z$

Fixed Points:

$\theta = 0 \quad z = \pm 1 \quad \left(\frac{m_1}{m_2} = 0, \infty\right)$

$\theta = \frac{\pi}{2} \quad z = \mp 1 \quad$ (symmetry)

$\theta = \frac{\pi}{4} \quad z = 0 \quad \left(\frac{m_1}{m_2} = 1,\ \phi = \frac{\pi}{4}\right)$

Stability:

For $x > 0$,

attractors $(\theta = 0,\ z = 1)$, $(\theta = \frac{\pi}{2},\ z = -1)$

repellors $(\theta = 0,\ z = -1)$, $(\theta = \frac{\pi}{2},\ z = +1)$

saddle pt. $(\theta = \frac{\pi}{4},\ z = 0)$

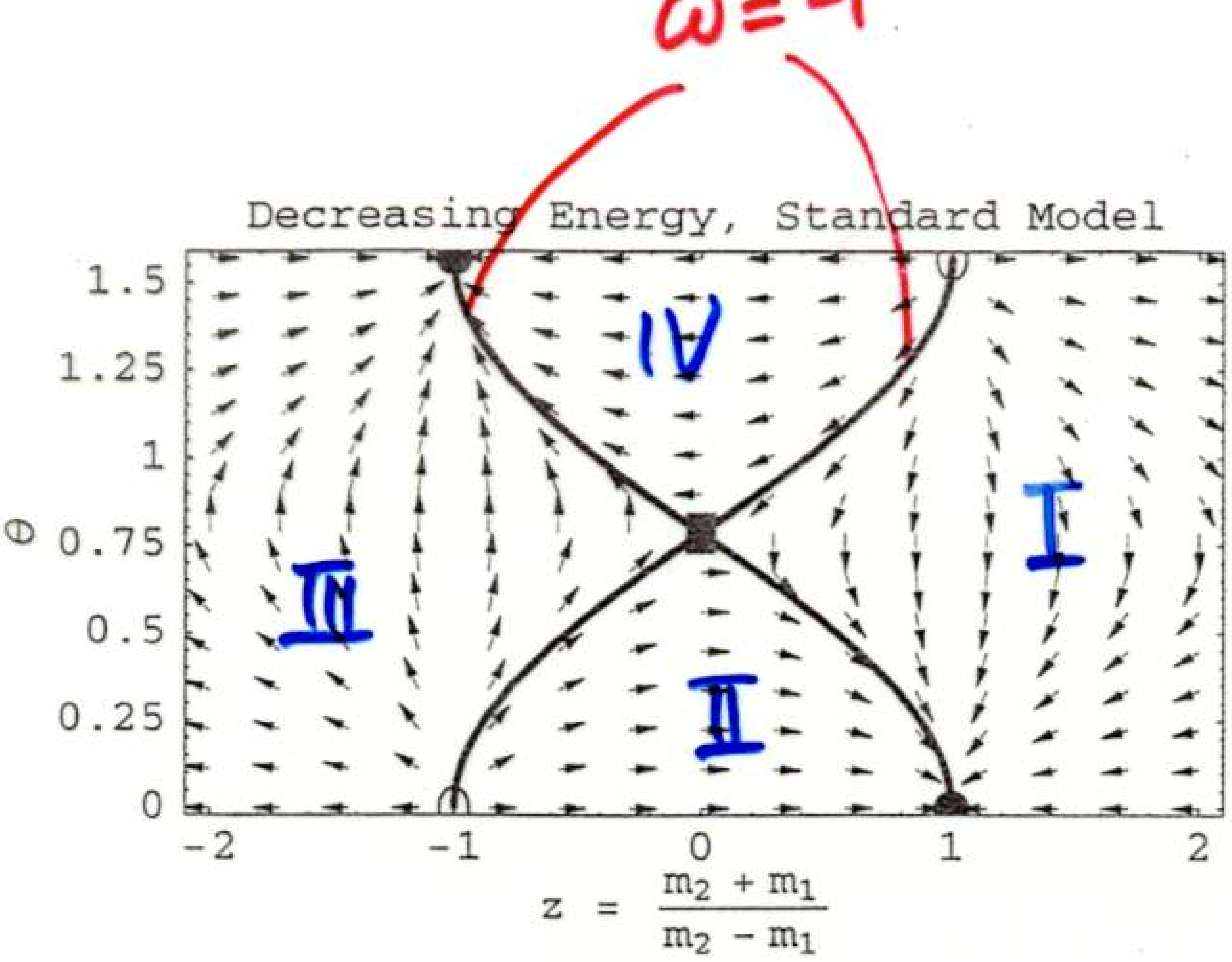
For $x < 0$,

attractor $\longleftrightarrow$ repellor

Trajectories:

integration of RGE,

$$\frac{\sin^2 2\theta}{z^2 - 1} = \omega = \text{const} = (\text{RGE inv.})^2$$




$\omega = -1$: trajectory ends at (flees from) $\theta = \frac{\pi}{4}$

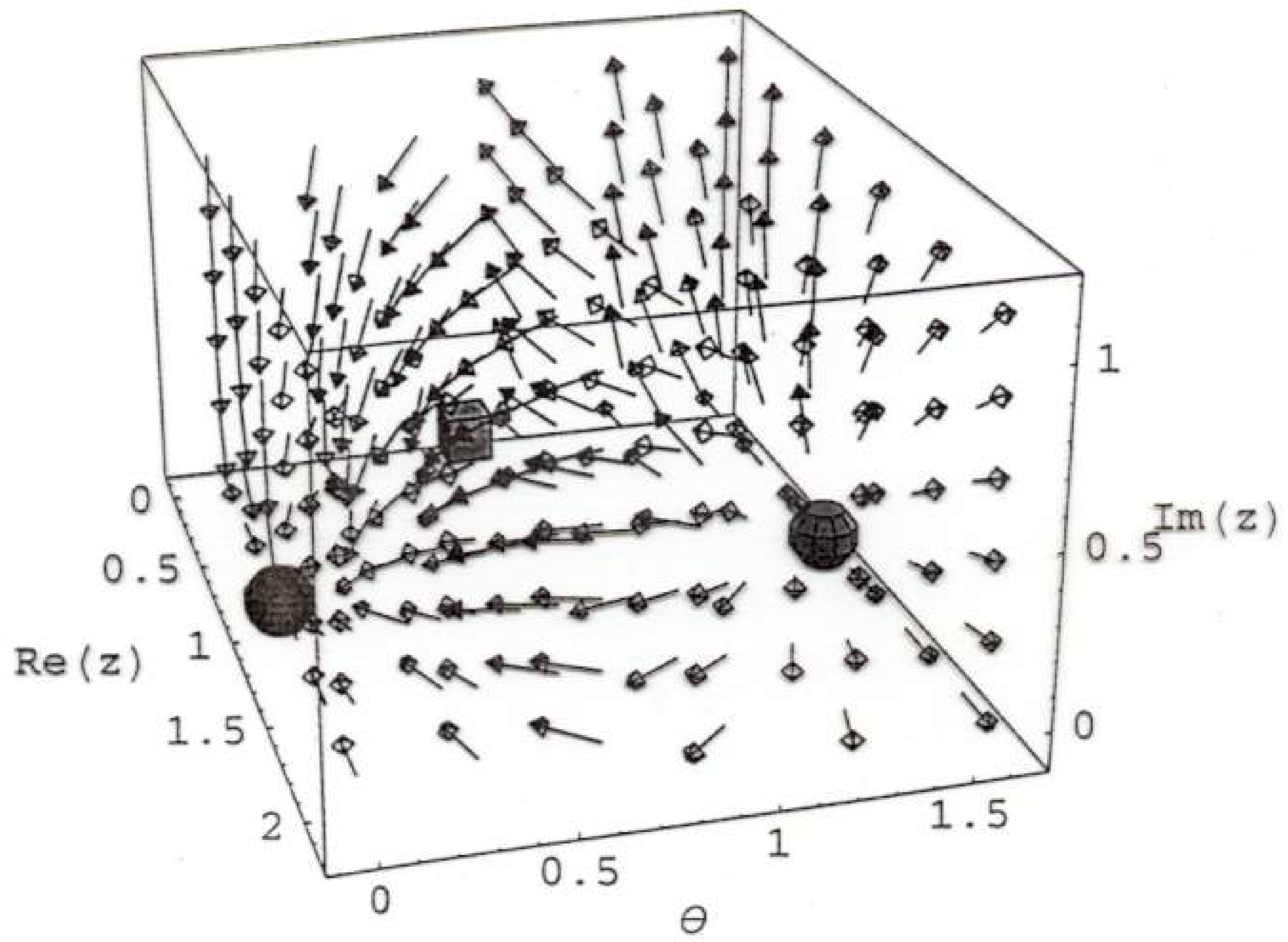
$\omega \geq 0$ (I): $\}$ pass through max. mixing
$\omega < -1$ (III)

$-1 < \omega < 0$ (II, IV): never attain max. mixing

symmetry is evident:

$$\theta \to \frac{\pi}{2} - \theta, \quad z \to -z$$

Figure 1: Phase portrait for the Standard Model RG equations when $\mathrm{Im}(z) = 0$. The different fixed points are shown with solid circles, open circles and grey square denoting attractors, repellors and saddle point. The solid curve shows the trajectories that connect the fixed points. The arrows are reversed for the MSSM.




• FP's confined to $\mathrm{Im}\, z = 0$ plane

• For SM, MSSM, $\theta = \frac{\pi}{4} \Leftrightarrow$ saddle pt.

Figure 2: The SM RGE evolution in $\mathrm{Re}(z)$, $\theta$, $\mathrm{Im}(z)$ space, for the range $0 < \mathrm{Re}(z)$, $0 < \mathrm{Im}(z)$, $0 < \theta < \pi/2$. The different fixed points are all confined to the $\mathrm{Im}(z) = 0$ plane. They are shown using a dark sphere, light sphere and a cuboid which denote the attractor, repellor and saddle point.

# Quantitative Analysis

- $|\xi| \ll 1$ . $\xi_{SM} \sim 10^{-5}$ $\xi_{MSSM} \lesssim O(10^{-2})$

## real mass matrices.

1) $\phi = \frac{\pi}{4}$ (opposite sign masses)

$\theta \simeq \theta_0$ no resonance (RGE inv. $S_{2\theta}\, ch2\eta = \text{const}$)

2) $\phi = 0$ (same sign masses)

$$\tan 2\theta = \frac{S_{2\theta_0}}{C_{2\theta_0}\, ch2\xi - \Sigma_R\, sh2\xi}\,, \quad \Sigma_R = \left(\frac{m_2+m_1}{m_2-m_1}\right)_0$$

max. mixing at

$$C_{2\theta_0} = \frac{\tanh 2\xi}{\tanh 2\eta_0}$$

possible if

1) $\xi$, $\eta_0$ same sign (assume $\theta_0 < \frac{\pi}{4}$)

2) $\eta_0 \sim \xi \ll 1$ for $C_{2\theta_0} \sim O(1)$

i.e. near-degeneracy

Introduce $\delta_0 = 1 - \left(\frac{m_1}{m_2}\right)_0$

max. mixing condition becomes

$$\boxed{C_{2\theta_0} \simeq \frac{4f}{\delta_0}}$$

RGE inv.

$$S_{2\theta}\,\mathrm{sh}2\eta = S_{2\theta_0}\,\mathrm{sh}2\eta_0$$

$\Rightarrow$ As $\theta$ is driven toward max. mixing $\theta = \frac{\pi}{4}$,

masses get more degenerate

# Complex Mass Matrices

- large RGE effect only if $\eta \ll 1$ (degenerate)
- condition for max. mixing

$$\boxed{\frac{\delta_0}{4\xi} \simeq \left( C_{2\theta_0} S^2_{2\phi_0} + \frac{1}{C_{2\theta_0}} C^2_{2\phi_0} \right) \quad \left(\phi_0 \neq \frac{\pi}{4}\right)}$$

$\Rightarrow$ $\frac{\delta_0}{4\xi} \sim O(1)$, degree of degeneracy

$\Longleftrightarrow$ RG evolution parameter $\xi$

$\Rightarrow$ peak position shifts with phase $\phi_0$

$\Rightarrow$ max. mixing possible for any phase except $\phi_0 = \frac{\pi}{4}$ (opposite sign masses)

- resonance width:

$$\Delta\left(\frac{m_1}{m_2}\right)_0 \simeq 8\xi C_{2\phi_0} \frac{S_{2\theta_0}}{C^2_{2\theta_0}} \sqrt{C^2_{2\phi_0} + C^2_{2\theta_0} S^2_{2\phi_0}}$$

$$\xrightarrow{\phi_0 \to \frac{\pi}{4}} 0$$

resonance (max. mixing) disappears at $\phi = \frac{\pi}{4}$ !

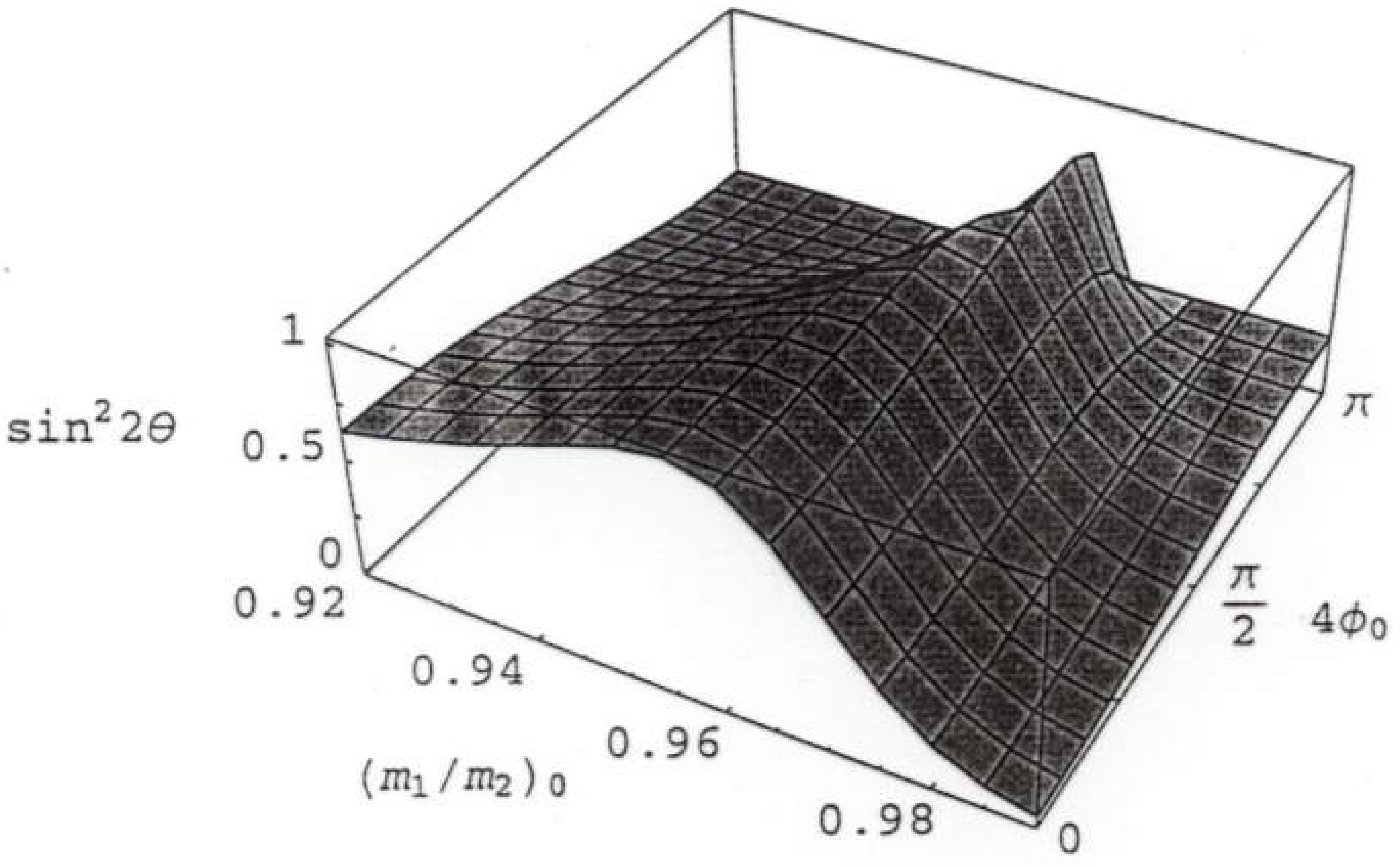



Figure 4: 3D plot for $\sin^2 2\theta$ vs. initial values $(m_1/m_2)_0$ and $4\phi_0$. The inputs are fixed at $\theta_0 = \pi/12$ and the RGE factor $\xi = 0.01$ (MSSM).

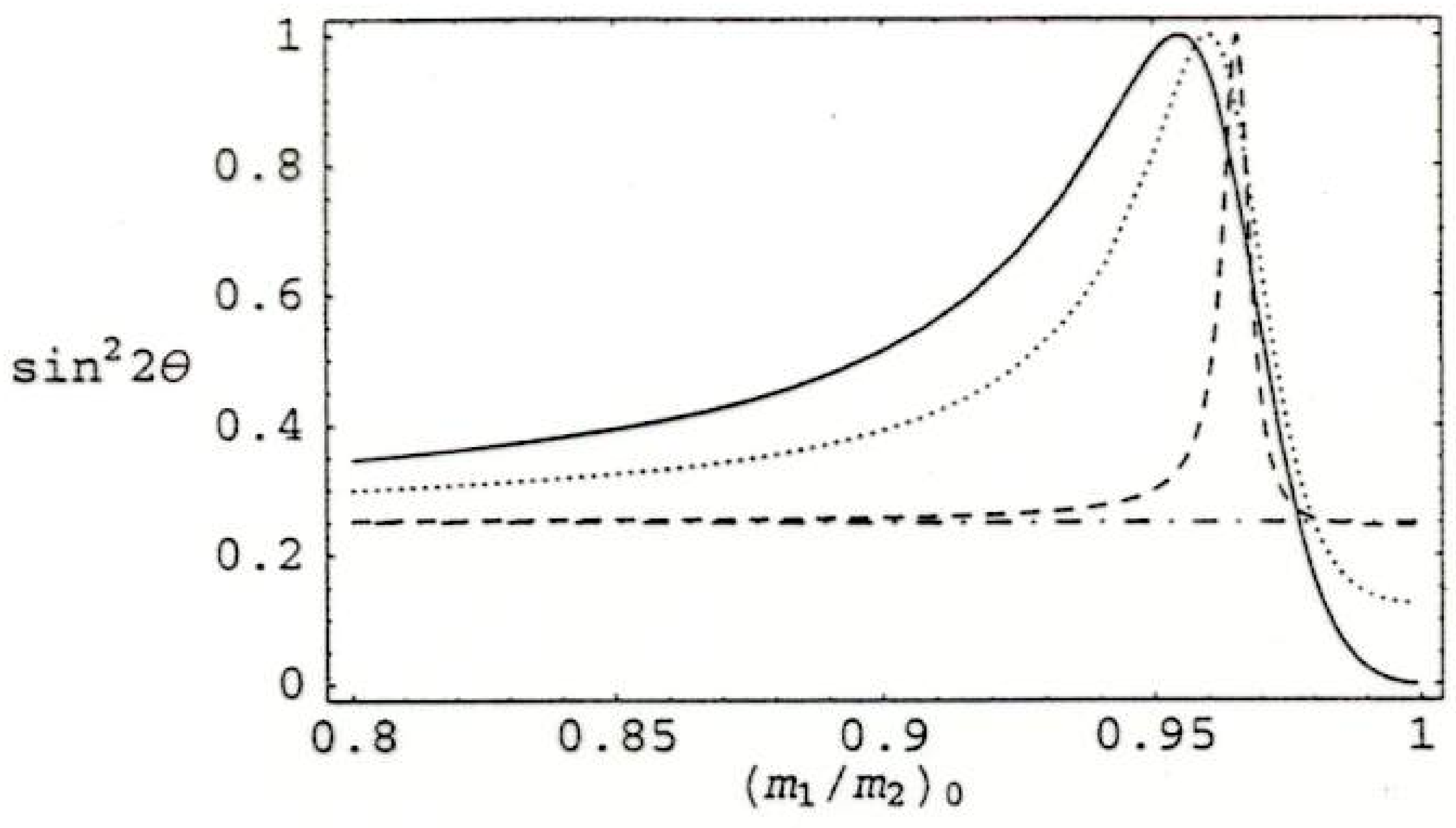



Figure 5: $\sin^2 2\theta$ vs. $(m_1/m_2)_0$ with inputs $\theta_0 = \pi/12$ and $\xi = 0.01$ (MSSM). The solid, dotted, dashed, and dot-dashed curves correspond to $4\phi_0 = 0, \pi/2, 0.9\pi$ and $\pi$ respectively.

- RGE inv.

$$s_{2\theta}\, sh2\eta\, c_{2\phi} = s_{2\theta_0}\, sh2\eta_0\, c_{2\phi_0}$$

$$s_{2\theta}\, ch2\eta\, s_{2\phi} = s_{2\theta_0}\, ch2\eta_0\, s_{2\phi_0}$$

near degeneracy. $\eta \ll 1$

$$s_{2\theta}\, s_{2\phi} = \text{const}.$$

anti-correlation between $(\theta, \varphi)$.

$\varphi$ is minimal at max. mixing.

generalization:

- For certain type III 2HDM, can have stable FP. (attractor) at $\theta = \frac{\pi}{4}$. then max. mixing from RGE evolution insensitive to initial condition
- 3 $\nu$, have 3 angles, 3 phases, 2 mass ratios, RGE inv. can be constructed using rephasing inv.

  $\Rightarrow$ 3 complex RGE inv.